DEDEKIND SUMS AND VALUES OF L-FUNCTIONS AT POSITIVE INTEGERS (Analytic Number Theory : related Multiple aspects of Arithmetic Functions)

BAYAD, ABDELMEJID

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ABDELMEJID BAYAD

ABSTRACT. In this paper, we study Dedekind sums and we connect them to the mean values of Dirichlet L-functions. For this, we introduce and investigate higher order dimensional Dedekind-Rademacher sums given by the expression

(1) $$S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}})=\frac{1}{a_0^{m_0+1}}\sum_{k=1}^{a_0-1}\prod_{j=1}^{d}\cot^{(m_j)}\left(\frac{\pi a_j k}{a_0}\right),$$

where $\overrightarrow{\mathbf{a_0}}=(a_0;a_1,\ldots,a_d),\overrightarrow{\mathbf{m_0}}=(m_0;m_1,\ldots,m_d)$, $a_0,a_1,\ldots,a_d$ are positive integers pairwise coprime and $m_0,m_1,\ldots,m_d$ are nonnegative integers. In this paper, we prove that the sums (1) are rational numbers, satisfy a Dedekind reciprocity type law, and their denominators have explicit and universal bounds. Our results recover and improve the well-known reciprocity and rationality theorems in [3, 13] and others. In connection with Dedekind sums we study the mean values of L-functions. For a given positive integer $q\geq 2$ and Dirichlet characters $\chi_1,\ldots,\chi_d \pmod q$, we investigate the mean value of the twisted product

$$\overline{\chi}_1(a_1)\cdots\overline{\chi}_d(a_d)L(m_1+1,\chi_1)\cdots L(m_d+1,\chi_d),$$

such that $m_1,\cdots,m_d$ have the same parity and

$$\chi_i(-1)=(-1)^{m_i+1},i=1,\cdots,d$$

as an application of our Dedekind reciprocity law, for the non twisted case we give explicit fromulae for this mean and we recover and improve the previous works of Walum [11], Louboutin Liu and Zhang [5, 6, 7, 14].

1. Higher dimensional Dedekind-Rademacher sums

Through this paper, for any $\overrightarrow{\mathbf{m}}=(m_0,\ldots,m_d)$ be a $(d+1)$-tuple of nonnegative integers, we denote by

$$|\overrightarrow{\mathbf{m}}|=\sum_{i=0}^{d}m_i,\ \overrightarrow{\mathbf{m}}!=\prod_{0\leq i\leq d}m_i!\ ,\ M=d+|\overrightarrow{\mathbf{m}}|\ .$$

Let us recall some definitions.

1.1. **Dedekind-Rademacher sums.** Let d,a_i be positive integers, $a_0,\ldots,\widehat{a_i},\ldots a_d$ are positive integers prime to a_i and $m_0,\ldots,m_d$ be non-negative integers. For $i=0,\ldots,d$, we consider the multiple Dedekind-Rademacher sum defined by

(2) $$S_d(\overrightarrow{\mathbf{a_i}},\overrightarrow{\mathbf{m_i}}):=\begin{cases}\dfrac{1}{a_i^{m_i+1}}\displaystyle\sum_{k=1}^{a_i-1}\prod_{\substack{j=0\\ j\neq i}}^{d}\cot^{(m_j)}\left(\frac{\pi a_j k}{a_i}\right) & \text{if } a_i\geq 2,\\ \\ 0 & \text{if } a_i=1,\end{cases}$$

where $\overrightarrow{\mathbf{a_i}}=(\mathbf{a}_i;a_0,\ldots,\widehat{a_i},\ldots,a_d)$, $\overrightarrow{\mathbf{m_i}}=(\mathbf{m}_i;m_0,\ldots,\widehat{m_i},\ldots,m_d)$ and as usual $\widehat{x_n}$ means we omit the term x_n . Throughout this paper , we set $M_i=d+\sum_{\substack{j\neq i\\ 0\leq j\leq d}}m_j$, and $\mathbb{N}$ denotes the set of nonnegative integers.

1.2. **Bernoulli functions.** The Bernoulli polynomials $B_k(x)$ are defined through the generating function

$$\frac{ze^{xz}}{e^z-1}=\sum_{k\geq 0}\frac{B_k(x)}{k!}z^k \tag{3}$$

and the Bernoulli numbers are $B_k := B_k(0)$. The Bernoulli functions $\bar{B}_k(x)$ are the periodized Bernoulli polynomials:

$$\bar{B}_k(x) := \begin{cases} 0 & \text{, if } x\in\mathbb{Z}, k=1; \\ B_k(\{x\}), & \text{otherwise.} \end{cases}$$

2. Statement of the results on Dedekind sums

2.1. Rationality theorem.

Theorem 2.1.1. *Let d, a_0 be positive integers, $a_1,\dots,a_d$ be positive integers prime to a_0, and $m_0,\dots,m_d$ be non-negative integers. We set $H=\frac{2^{M-m_0}}{i^{(M-m_0)}(m_1+1)\cdots(m_d+1)}$. Then we have*

$$a_0^{m_0+1}H^{-1}S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}})=a_0^{M-m_0-d+1}\sum_{\substack{0\leq n_1,\dots,n_d\leq a_0-1\\ a_0|n_1a_1+\cdots+n_da_d}}\overline{B}_{m_1+1}\left(\frac{n_1}{a_0}\right)\cdots\overline{B}_{m_d+1}\left(\frac{n_d}{a_0}\right)$$
$$-\overline{B}_{m_1+1}(0)\cdots\overline{B}_{m_d+1}(0).$$

Remarks 2.1.2. *Since the coefficients of Bernoulli polynomials $B_n(x)$ are rationals, then the sum $S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}})$ is a rational number. The denominator of this rational number is given by the Theorem 2.5.1 below.*

2.2. **Proof of the Theorem 2.1.1.** We use the well-known lemma.

Lemma 2.2.1. *Let m be a non-negative integer, a be an integer ≥ 2 and k be an integer not divisible by a. Then we have*

$$\overline{B}_m(x)=-\frac{m!}{(2\pi i)^m}\sum_{\ell\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi i\ell x}}{\ell^m} \tag{4}$$

and

$$\cot^{(m-1)}\left(\frac{\pi k}{a}\right)=\frac{1}{ma}\left(\frac{2a}{i}\right)^m\sum_{n=0}^{a-1}e^{-2\pi ikn/a}\overline{B}_m\left(\frac{n}{a}\right).$$

To use this lemma, we set

$$A=\frac{1}{(m_1+1)\cdots(m_d+1)a_0^d}\left(\frac{2a_0}{i}\right)^{M-m_0}.$$

Then, we have

$$\begin{aligned} a_0^{m_0+1}S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}}) &= A\sum_{t=1}^{a_0-1}\prod_{j=1}^{d}\sum_{n_j=0}^{a_0-1}\exp\left(\frac{-2\pi i n_j t a_j}{a_0}\right)\overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right)\\ &= A\sum_{t=1}^{a_0-1}\sum_{0\leq n_1,\dots,n_d\leq a_0-1}\exp\left(\frac{-2\pi it}{a_0}\left(\sum_{j=1}^{d}n_ja_j\right)\right)\prod_{j=1}^{d}\overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right). \end{aligned}$$

Since

$$\sum_{t=1}^{a_0-1}\exp\left(\frac{-2\pi it}{a_0}\left(\sum_{j=1}^{d}n_ja_j\right)\right)=\begin{cases} a_0-1 & \text{if } a_0|n_1a_1+\cdots+n_da_d\\ \\ -1 & \text{otherwise,}\end{cases}$$

it follows that

$$\begin{aligned} a_0^{m_0+1} S_d(\overrightarrow{\mathbf{a_0}}, \overrightarrow{\mathbf{m_0}}) &= A\Big(\sum_{\substack{0\le n_1,\dots,n_d\le a_0-1 \\ a_0 | n_1a_1+\cdots+n_da_d}} (a_0-1)\prod_{j=1}^d \overline{B}_{m_j}\left(\frac{n_j}{a_0}\right) - \sum_{\substack{0\le n_1,\dots,n_d\le a_0-1 \\ a_0 \nmid n_1a_1+\cdots+n_da_d}} \prod_{j=1}^d \overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right)\Big) \\ &= A\Big(a_0 \sum_{\substack{0\le n_1,\dots,n_d\le a_0-1 \\ a_0 | n_1a_1+\cdots+n_da_d}} \prod_{j=1}^d \overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right) - \sum_{0\le n_1,\dots,n_d\le a_0-1} \prod_{j=1}^d \overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right)\Big). \end{aligned}$$

Finally note that this last sum is equal to

$$\prod_{j=1}^d \sum_{n_j=0}^{a_0-1} \overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right) = \prod_{j=1}^d a_0^{-m_j} \overline{B}_{m_j+1}(0) = a_0^{-m_1-\cdots-m_d} \prod_{j=1}^d \overline{B}_{m_j+1}(0)$$

where we have used the classical Raabe formula [8].
This completes the proof of Theorem 2.1.1.

2.3. Dedekind Reciprocity Law.

Next we state the reciprocity law for these sums that allows us to compute them.

Theorem 2.3.1 ([1]). *Let d be a positive integer, $a_0, \dots, a_d$ be pairwise coprime positive integers and $\overrightarrow{\mathbf{m}} = (m_0, \dots, m_d)$ be a $(d+1)$-tuple of non-negative integers. Assume that $M = d + |\overrightarrow{\mathbf{m}}|$ is even. Then we have*

$$\sum_{i=0}^d (-1)^{m_i} m_i! \sum_{(\ell_0,\cdots,\ell_d)}^{*i} \Big(\prod_{\substack{j=0\\ j\ne i}}^d \frac{a_j^{\ell_j}}{\ell_j!}\Big) S_d(\overrightarrow{\mathbf{a_i}}, \overrightarrow{\mathbf{m_i}} + \overrightarrow{\mathbf{L_i}}) = \begin{cases} R + (-1)^{d/2} & \text{, if all } m_i \text{ are zero;} \\ R & \text{, otherwise} \end{cases}$$

where $\sum^{*i}$ denotes summation over all $\ell_0, \dots, \widehat{\ell_i}, \dots, \ell_d \ge 0$ such that

$$|\overrightarrow{\mathbf{L_i}}| = m_i,\ \overrightarrow{\mathbf{L_i}} = (\ell_i; \ell_0, \dots, \widehat{\ell_i}, \dots, \ell_d)$$

and

(5) $$R = \frac{(-1)^{M/2} 2^M}{\prod_{i=0}^d a_i^{m_i+1}} \sum_{\substack{j_0,\dots,j_d\ge 0\\ j_0+\cdots+j_d=M/2}} \prod_{i=0}^d a_i^{2j_i} A_{i,j_i}.$$

and

$$A_{i,j_i} = \begin{cases} \frac{B_{2j_i}}{(2j_i-1-m_i)!(2j_i)} & \text{if } j_i \text{ is an integer } \ge (m_i+1)/2, \\ (-1)^{m_i} m_i! & \text{if } j_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example. When all m_i **are zero**, we have $M = d$ and $A_{i,j_i} = \frac{(-1)^{j_i} 2^{2j_i} B_{2j_i}}{(2j_i)!}$, hence the right member of the reciprocity formula in Theorem 2.3.1 becomes

(6) $$R + (-1)^{d/2} = (-1)^{d/2}\Big(1 - \frac{2^d}{a_0 \dots a_d} \sum_{\substack{j_0,\dots,j_d\ge 0\\ j_0+\cdots+j_d=d/2}} \prod_{i=0}^d \frac{B_{2j_i}}{(2j_i)!} a_i^{2j_i}\Big).$$

2.4. **Proof of the reciprocity Theorem 2.3.1.** Let us consider the function f of the complex variable z defined by

$$f(z) = \prod_{j=0}^{d} \cot^{(m_j)}(\pi a_j z).$$

Let ε be a fixed real number with $\varepsilon \in]0, \min_{0\leq j\leq d} 1/a_j[$. Let $y > 0$ be a real parameter. We set $A = (1-\varepsilon) + yi$, $B = -\varepsilon + yi$, $C = \overline{B}$ and $D = \overline{A}$, and we consider the rectangular path $\gamma := [A, B, C, D, A]$. We want to integrate f along γ by applying Cauchy's Residue Theorem. The poles of f lying inside γ are:

• the point $z_0 = 0$ which is a pole of order $M+1$;

• the points k_j/a_j, where $k_j = 1, \ldots, a_j - 1$, $a_j \neq 1$ and $j = 0, \ldots, d$, which are distinct since the integers a_j are pairwise coprime. Every point k_j/a_j is a pole of f of order $(m_j + 1)$. By Cauchy's Residue Theorem, we have

$$\frac{1}{2\pi i}\int_\gamma f(z)\underset{\cdot}{z} = \operatorname{Res}(f,0) + \sum_{j=0}^{d}\sum_{k=1}^{a_j-1} \operatorname{Res}(f, k/a_j).$$

Since 1 is a period of f, we see that

$$\int_{[D,A]} f(z)\underset{\cdot}{z} = -\int_{[B,C]} f(z)\underset{\cdot}{z}.$$

Furthermore, setting $\delta = \pm 1$, we have for all real t

$$\lim_{y\to+\infty} \cot^{(m)}(t+\delta yi) = \begin{cases} -\delta i & \text{if } m = 0,\\ 0 & \text{if } m \geq 1.\end{cases}$$

Hence

$$\int_\gamma f(z)\underset{\cdot}{z} = \quad = \begin{cases} 2i^{d+1} & \text{if all } m_i \text{ are zero and } d \text{ is even},\\ 0 & \text{otherwise.}\end{cases}$$

• Note that if $a_j = 1$ the sum over k is equal to 0.
Therefore, we obtain

$$\sum_{j=0}^{d}\sum_{k=1}^{a_j-1} \operatorname{Res}(f, k/a_j) = \begin{cases} -\operatorname{Res}(f,0) + i^d/\pi & \text{if all } m_i \text{ are zero and } d \text{ is even},\\ -\operatorname{Res}(f,0) & \text{otherwise.}\end{cases} \tag{7}$$

Now, we need to evaluate the two sides of (7).

1. Residue of f at $z = 0$. The Laurent expansion of the cotangent at 0:

$$\cot(w) = \frac{1}{w} + \sum_{j=1}^{+\infty} \frac{(-1)^j 2^{2j} B_{2j}}{(2j)!} w^{2j-1} \qquad (0 < |w| < \pi)$$

implies

$$w^{m+1}\cot^{(m)}(w) \quad = \quad (-1)^m m! + \sum_{\substack{j \text{ integer}\\ 2j-m\geq 1}} \frac{(-1)^j 2^{2j-1} B_{2j}}{(2j-1-m)!j} w^{2j}$$

for every integer $m \geq 1$.

For $i \in \{0, \dots, d\}$, let us set

$$(8) \qquad A_{i,j_i} = \begin{cases} \frac{B_{2j_i}}{(2j_i-1-m_i)!(2j_i)} & \text{if } j_i \text{ integer } \geq (m_i+1)/2, \\ (-1)^{m_i} m_i! & \text{if } j_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So, we have

$$\begin{aligned} \operatorname{Res}(f,0) &= \pi^{-M-1} \prod_{i=0}^{d} a_i^{-m_i-1} \sum_{\substack{(j_0,\dots,j_d)\in\mathbb{N}^{d+1} \\ j_0+\dots+j_d=M/2}} \prod_{i=0}^{d} (-1)^{j_i} (2\pi a_i)^{2j_i} A_{i,j_i} \\ &= \frac{(-1)^{M/2} 2^M}{\pi \prod_{i=0}^{d} a_i^{m_i+1}} \sum_{\substack{(j_0,\dots,j_d)\in\mathbb{N}^{d+1} \\ j_0+\dots+j_d=M/2}} \prod_{i=0}^{d} a_i^{2j_i} A_{i,j_i}. \end{aligned}$$

2. Residue of f at the other poles. For any integer $a_i > 1$ and $1 \leq k \leq a_i - 1$, we have

$$\operatorname{Res}(f, k/a_i) = (-1)^{m_i} \frac{m_i!}{a_i^{m_i+1}\pi} \sum_{\substack{(\ell_0,\dots,\widehat{\ell_i},\dots,\ell_d)\in\mathbb{N}^d \\ \ell_0+\dots+\widehat{\ell_i}+\dots+\ell_d=m_d}} \prod_{\substack{j=0 \\ j\neq i}}^{d} \frac{a_j^{\ell_j}}{\ell_j!} \cot^{(m_j+\ell_j)}\left(\frac{\pi k a_j}{a_i}\right).$$

Consequently, we have obtained the relation

$$\sum_{\substack{i=0 \\ a_i\neq 1}}^{d} \sum_{k=1}^{a_i-1} \operatorname{Res}(f, k/a_i) = \frac{1}{\pi} \sum_{i=0}^{d} (-1)^{m_i} \frac{m_i!}{a_i^{m_i+1}} \sum_{\substack{(\ell_0,\dots,\widehat{\ell_i},\dots,\ell_d)\in\mathbb{N}^d \\ \ell_0+\dots+\widehat{\ell_i}+\dots+\ell_d=m_i}} \Big(\prod_{\substack{j=0 \\ j\neq i}}^{d} \frac{a_j^{\ell_j}}{\ell_j!}\Big) \sum_{k=1}^{a_i-1} \prod_{\substack{j=0 \\ j\neq i}}^{d} \cot^{(m_j+\ell_j)}\left(\frac{\pi k a_j}{a_i}\right).$$

2.5. **Universal Bounds.**

In the following theorem we study the universal bound for the denominator of the higher order dimensional Dedekind sums.

Theorem 2.5.1. *Let d, a_0 be positive integers, $a_0, a_1, \dots, a_d$ be positive integers relatively prime to a_0 and $m_0, \dots, m_d$ be non-negative integers. We set*

$$\mu := \prod_{\substack{3\leq p\leq M+1 \\ p \text{ prime}}} p^{\left[\frac{M}{p-1}\right]}, \quad \Delta := \gcd\Big(\mu\,;\, a_0^{d-1}(m_1+1)\cdots(m_d+1) \prod_{j=1}^{d} \prod_{\substack{p\leq m_j \\ p \text{ prime} \geq 3}} p\Big).$$

Then we have

$$a_0 S_d(\overrightarrow{\mathbf{a_0}}, \overrightarrow{\mathbf{m_0}}) \in \frac{2^{m_1+\dots+m_d}}{\Delta}\mathbb{Z}.$$

Remark 7. The reason, that we are interested in μ and Δ is that these are the **universal bounds** for the denominator of our higher order dimensional Dedekind sums. For any d, a_0 be a positive integer, $a_1, \dots, a_d$ be positive integers prime to a_0 we obtain

$$a_0 \Delta S_d(\overrightarrow{\mathbf{a_0}}, \overrightarrow{\mathbf{m_0}}) \in 2^{m_1+\dots+m_d}\mathbb{Z}.$$

For instance, if $m_0 = \dots = m_d = 0$, we obtain $a_0 \mu S_d(\overrightarrow{\mathbf{a_0}}, \overrightarrow{\mathbf{m_0}}) \in \mathbb{Z}$, this is the rationality theorem of Zagier [13, p.160]. Our method gives us a simple and new way to get this theorem of Zagier.

2.6. **Proof of the universal bound Theorem 2.5.1.** For the classical von Staudt-Clausen theorem we can see [4, 9, 12]. For any non negative integer m and any prime number p, let $v_p(m)$ denote the p-adic valuation of m. We have the useful lemma.

Lemma 2.6.1. *Let n be an integer ≥ 1. We denote by D_n the denominator of $\frac{B_n}{n!}$. For any prime p, we have*

$$v_p(D_n) \leq \left[\frac{n}{p-1}\right]. \tag{9}$$

Proof. Using the classical von Staudt theorem, it's easy to see that

$$v_p(\text{denominator of } B_n) = \begin{cases} 1 & \text{, if } p-1 | n; \\ 0 & \text{, otherwise.} \end{cases}$$

On the other hand, we have the well known fact. For every prime number p

$$v_p(n!) \leq \begin{cases} [n/(p-1)] & \text{, if } p-1 \text{ does not divide } n; \\ \\ [n/(p-1)] - 1 & \text{, if } p-1 \text{ divides } n. \end{cases}$$

This yields the desired lemma.

Proof of Theorem 2.5.1. From the Theorem 2.1.1 we study the denominator of $a_0^{m_0+1} S_d(\overrightarrow{\mathbf{a_0}}, \overrightarrow{\mathbf{m_0}})$. Let D' be the denominator of $\displaystyle\sum_{\substack{0 \leq n_1, \ldots, n_d \leq a_0 - 1 \\ a_0 | n_1 a_1 + \cdots + n_d a_d}} \overline{B}_{m_1+1}\left(\frac{n_1}{a_0}\right) \cdots \overline{B}_{m_d+1}\left(\frac{n_d}{a_0}\right)$.

Then $D' | D_1 \cdots D_d$ where D_j is the denominator of $\overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right)$ $(j = 1, \ldots, d)$. If $(m_j, n_j) \neq (0, 0)$, we have

$$\begin{aligned} \overline{B}_{m_j+1}\left(\frac{n_j}{a_0}\right) = B_{m_j+1}\left(\frac{n_j}{a_0}\right) &= \sum_{k=0}^{m_j+1} \binom{m_j+1}{k} \left(\frac{n_j}{a_0}\right)^{m_j+1-k} B_k \\ &= \frac{1}{a_0^{m_j+1}} \left(n_j^{m_j+1} + \sum_{k=1}^{m_j+1} \binom{m_j+1}{k} a_0^k n_j^{m_j+1-k} B_k\right). \end{aligned}$$

By von Staudt's Theorem, we know that if k is even, the denominator of B_k is $\displaystyle\prod_{\substack{p \text{ prime} \\ p-1|k}} p$, and therefore

$$D_j | a_0^{m_j+1} \prod_{\substack{p \leq m_j+2 \\ p \text{ prime}}} p \qquad (j = 1, \ldots, d). \tag{10}$$

Thus we obtain

$$D' | 2^d a_0^{\sum_{j=1}^d (m_j+1)} \prod_{j=1}^d \prod_{\substack{p \leq m_j+2 \\ p \text{ prime } \geq 3}} p. \tag{11}$$

Furthermore, if D'' is the denominator of $\overline{B}_{m_1+1}(0) \cdots \overline{B}_{m_d+1}(0)$, all $m_j \neq 0$ and all m_{j+1} are even, then we have

$$D'' | 2^d \prod_{\substack{p \leq m_j+2 \\ p \text{ prime } \geq 3}} p. \tag{12}$$

So it follows that

$$(13)\qquad a_0^{m_0+1}S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}})=\frac{2^{m_1+\cdots+m_d}N_0}{(m_1+1)\cdots(m_d+1)a_0^{d-1}\prod_{j=1}^{d}\prod_{\substack{p\le m_j+2\\ p\text{ prime }\ge 3}}p},$$

where $N_0\in\mathbb{Z}$.

Obviously, (13) can be written as

$$a_0^{d+m_0}\frac{(m_1+1)\dots(m_d+1)}{2^{m_1+\cdots+m_d}}\Big(\prod_{j=1}^{d}\prod_{\substack{p\le m_j+2\\ p\ge 3\text{ prime}}}p\Big)S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}})\in\mathbb{Z}.$$

End of the proof of Theorem 2.5.1. We shall now apply Theorem 2.3.1. We begin by giving the denominator of the rational number R defined by (5). Write

$$\prod_{i=0}^{d}a_i^{2j_i}A_{i,j_i}\;=\;\frac{A}{(2j_0)!\cdots(2j_d)!}B_{2j_0}\cdots B_{2j_d}$$

where $A\in\mathbb{Z}$ and denote by D the denominator of this rational number. By Lemma 2.6.1, we have for all prime numbers p,

$$v_p(D)\le\sum_{i=0}^{d}\left[\frac{2j_0}{p-1}\right]\le\left[\sum_{i=0}^{d}\frac{2j_0}{p-1}\right]=\left[\frac{M}{p-1}\right].$$

Let

$$\mu:=\prod_{\substack{2<p\le M+1\\ p\text{ prime}}}p^{\left[\frac{M}{p-1}\right]}.$$

It follows that

$$(14)\qquad D|2^M\mu.$$

It is then easy to deduce that the number R defined by (5) can be written as

$$R=\frac{N_1'}{\mu\prod_{i=0}^{d}a_i^{m_i+1}}\qquad(N_1'\in\mathbb{Z}).$$

Therefore, by Theorem 2.3.1 we can write

$$(15)\sum_{i=0}^{d}(-1)^{m_i}m_i!\sum_{\substack{\ell_0,\dots,\widehat{\ell_i},\dots,\ell_d\ge 0\\ \ell_0+\cdots+\widehat{\ell_i}+\cdots+\ell_d=m_i}}\prod_{\substack{j=0\\ j\ne i}}^{d}\frac{a_j^{\ell_j}}{\ell_j!}S_d(\overrightarrow{\mathbf{a_i}},\overrightarrow{\mathbf{m_i}}+\overrightarrow{\mathbf{L_i}})=\frac{N}{\mu\prod_{i=0}^{d}a_i^{m_i+1}},\quad(N\in\mathbb{Z}).$$

If we apply a formula similar to (13), we can write for some $N_i\in\mathbb{Z}$

$$\begin{aligned}(16)\qquad & a_i^{m_i+1}S_d(\overrightarrow{\mathbf{a_i}},\overrightarrow{\mathbf{m_i}}+\overrightarrow{\mathbf{L_i}})\\ &=\frac{2^{\sum_{j\ne i}m_j+\ell_j}N_i}{a_i^{d-1}\prod_{j\ne i}(m_j+\ell_j+1)\prod_{j\ne i}\prod_{\substack{p\le m_j+\ell_j\\ p\text{ prime }\ge 3}}p}\\ &=\frac{2^{m_0+\cdots+m_d}N_i}{a_i^{d-1}\prod_{j\ne i}(m_j+\ell_j+1)\prod_{j\ne i}\prod_{\substack{p\le m_j+\ell_j\\ p\text{ prime }\ge 3}}p}\end{aligned}$$

under the condition $\sum_{\substack{j=0\\ j\neq i}}^{d} \ell_j = m_i$.

We note that $\mu / \prod_{\substack{j=0\\ j\neq i}}^{d} \prod_{\substack{p\le m_j+\ell_j\\ p\ge 3 \text{ prime}}} p \in \mathbb{N}$, because $m_j + \ell_j \le M+1$ and the number of j such that $p \le m_j + \ell_j$ is less than $\left[\frac{1}{p}\sum_{j\neq i} m_j + \ell_j\right] = \left[\frac{M-d}{p}\right]$. Therefore, we can write the quantity in (15) as follows

$$2^{m_0+\cdots+m_d}\sum_{i=0}^{d}(-1)^{m_i}m_i! \sum_{\substack{\ell_0,\ldots,\widehat{\ell_i},\ldots,\ell_d\ge 0\\ \ell_0+\cdots+\widehat{\ell_i}+\cdots+\ell_d=m_i}} \prod_{\substack{j=0\\ j\neq i}}^{d} \frac{a_j^{m_j+1+\ell_j}}{\ell_j!} \frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{d}(m_j+\ell_j+1)} \frac{\mu}{\prod_{\substack{j=0\\ j\neq i}}^{d} \prod_{\substack{p\le m_j+\ell_j\\ p \text{ prime} \ge 3}} p} \frac{N_i}{a_i^{d-1}} \in \mathbb{Z}.$$

This gives

$$2^{m_0+\cdots+m_d}\sum_{i=0}^{d}(-1)^{m_i}m_i! \sum_{\substack{\ell_0,\ldots,\widehat{\ell_i},\ldots,\ell_d\ge 0\\ \ell_0+\cdots+\widehat{\ell_i}+\cdots+\ell_d=m_i}} \prod_{\substack{j=0\\ j\neq i}}^{d} \frac{a_j^{m_j+1+\ell_j}}{\ell_j!} \frac{T}{\prod_{\substack{j=0\\ j\neq i}}^{d}(m_j+\ell_j+1)} \frac{\mu}{\prod_{\substack{j=0\\ j\neq i}}^{d} \prod_{\substack{p\le m_j+\ell_j\\ p \text{ prime} \ge 3}} p} \frac{N_i}{a_i^{d-1}} \in T\mathbb{Z}$$

where T is the least common multiple of the numbers

$$\prod_{\substack{j=0\\ j\neq i}}^{d}(m_j+\ell_j+1) : i = 0,\ldots,d; \ell_0+\cdots+\widehat{\ell_i}+\cdots+\ell_d = m_i.$$

Since the integers a_i are pairwise coprime, it follows that for each $i = 0,\ldots,d$

$$2^{m_0+\cdots+m_d}\frac{m_i!}{\prod_{\substack{j=0\\ j\neq i}}^{d}\ell_j!} \frac{T}{\prod_{\substack{j=0\\ j\neq i}}^{d}(m_j+\ell_j+1)} \frac{\mu}{\prod_{\substack{j=0\\ j\neq i}}^{d} \prod_{\substack{p\le m_j+\ell_j\\ p \text{ prime} \ge 3}} p} \frac{N_i}{a_i^{d-1}} \in T\mathbb{Z}.$$

For instance, if $i = 0$ we have

$$a_0^{d-1} | \frac{N_0}{\prod_{j=1}^{d}(m_j+1)} \frac{\mu}{\prod_{j=1}^{d} \prod_{\substack{p\le m_j\\ p \text{ prime} \ge 3}} p} \quad \text{since } m_0 = 0 \text{ and } \ell_1 = \cdots = \ell_d = 0. \tag{17}$$

By (13) and (17) we thus arrive at

$$\begin{cases} a_0^{d-1}(m_1+1)\ldots(m_d+1)(\prod_{j=1}^{d}\prod_{\substack{p\le m_j\\ p\ge 3}} p)a_0 S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}}) \in 2^{m_1+\cdots+m_d}\mathbb{Z} \\ \mu a_0 S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}}) \in 2^{m_1+\cdots+m_d}\mathbb{Z}. \end{cases} \tag{18}$$

This clearly implies the desired result that

$$a_0 S_d(\overrightarrow{\mathbf{a_0}},\overrightarrow{\mathbf{m_0}}) \in \frac{2^{m_1+\cdots+m_d}}{\Delta}\mathbb{Z}, \tag{19}$$

where $\Delta = \gcd\left(\mu\,;\, a_0^{d-1}(m_1+1)\dots(m_d+1)\prod_{j=1}^{d}\prod_{\substack{p\le m_j\\ p\text{ prime}\ge 3}} p\right)$.

3. Twisted mean values of L-functions: Introduction

Let q be a positive integer ≥ 2 and χ be a character modulo q, and $L(s,\chi)$ be the Dirichlet L-function corresponding to χ:

$$L(s,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^s},$$

where $\Re e\,(s)>0$ if χ is non principal and $\Re e\,(s)>1$ if χ is the principal character. Let $m_1,\dots,m_d$ be non negative integers. We shall here be interested by the study of the mean values

$$\sum_{(\chi_1,\cdots,\chi_d)}^{*}\prod_{i=1}^{d}\overline{\chi}_i(a_i)L(m_i+1,\chi_i),$$

where $\sum^*$ denotes summation over all characters $\chi_1,\dots,\chi_d \pmod q$ such that:

$$\chi_1\cdots\chi_d = 1\,,\chi_1(-1)=\cdots=\chi_d(-1)=(-1)^{m_1+1}=\cdots=(-1)^{m_d+1}.$$

Its well-known that in the case $d=2, m_1=m_2=0$ and $\chi_2=\overline{\chi}_1$, Walum [11] showed that for prime $q=p\geq 3$, the explicit formula

$$\sum_{\substack{\chi_1 \pmod p\\ \chi_1(-1)=-1}}|L(1,\chi_1)|^2=\frac{\pi^2(p-1)^2(p-2)}{12p^2}. \tag{20}$$

This result has extended by Louboutin [5] and Zhang [14] to any positive integer $q\geq 2$ by the formula as follows

$$\sum_{\substack{\chi_1 \pmod q\\ \chi_1(-1)=-1}}|L(1,\chi_1)|^2=\frac{\pi^2}{12}\frac{\varphi^2(q)}{q^2}\Big(q\prod_{\substack{p|q\\ p\text{ prime}}}(1+\frac{1}{p})-3\Big) \tag{21}$$

where $\varphi(q)$ is the Euler function. Moreover, Louboutin [6] has considered the case $d=2, m_1 = m_2 = k$ and proved the formula

$$\frac{2}{\varphi(q)}\sum_{\substack{\chi_1 \pmod q\\ \chi_1(-1)=-1}}|L(k,\chi_1)|^2=\frac{(2\pi)^{2k}}{2((k-1)!)^2}\sum_{l=0}^{2k}r_{k,l}\varphi_l(q)q^{l-2k}, \tag{22}$$

where

$$\varphi_l(q):=\prod_{\substack{p|q\\ p\text{ prime}}}\left(1-\frac{1}{p^l}\right),$$

and the coefficients $r_{k,l}$ are real numbers that were not given explicitly. In 2006, Liu and Zhang [7] treated the mean values of $L(m,\chi_1)L(n,\overline{\chi}_1)$ at positive integers $m,n\geq 1$,

$$\frac{2}{\varphi(q)}\sum_{\substack{\chi_1 \pmod q\\ \chi_1(-1)=-1}}L(m,\chi_1)L(n,\overline{\chi}_1)= \tag{23}$$

$$\frac{(-1)^{\frac{m-n}{2}}(2\pi)^{m+n}}{2(m!n!)}\left(\sum_{l=0}^{m+n}r_{m,n,l}\varphi_l(q)q^{l-m-n}-\frac{\epsilon_{m,n}}{q}B_mB_n\varphi_{m+n-1}(q)\right).$$

where

$$r_{m,n,l} = B_{m+n-l}\sum_{a=0}^{m}\sum_{b=0}^{n} B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1}.$$

4. Statement of results on mean values of L-functions

We have the interesting results

Theorem 4.0.2. *Let d be an integer ≥ 1 and $\vec{\mathbf{m}}= (m_1,\ldots,m_d)$ a d-tuple of positive integers such that $d+\mid \vec{\mathbf{m}}\mid$ is even. Let q be an integer ≥ 2. Let $a_1,\ldots,a_{d-1}$ be positive such that $(a_i,q)=1$ $(i=1,\ldots,d-1)$. We set $a_d=1$. Then we have*

$$\sum_{(\chi_1,\ldots,\chi_d)}^{*}\prod_{i=1}^{d}\overline{\chi}_i(a_i)L(m_i+1,\chi_i) = A_q(\vec{\mathbf{m}})\sum_{\substack{b\mid q\\ b\neq 1}} b\mu\left(\frac{q}{b}\right)S_d(\vec{\mathbf{a_0}},\vec{\mathbf{m_0}})$$

where

$$A_q(\vec{\mathbf{m}}) = \frac{(-1)^d}{2^d(\vec{\mathbf{m}}!)}(\frac{\pi}{q})^M\varphi(q)^{d-1},\ \vec{\mathbf{a_0}}= (b;a_1,\ldots,a_d),\ \vec{\mathbf{m_0}}= (0;m_1,\ldots,m_d)$$

The above theorem gives immediately

Corollary 4.0.3. *Let m and n be positive having same parity. Let a be a positive integer such that $(a,q)=1$. Then we have*

$$\sum_{\substack{\chi \pmod q\\ \chi(-1)=(-1)^{m+1}}}\overline{\chi}(a)L(m+1,\chi)L(n+1,\overline{\chi}) = A\sum_{\substack{b\mid q\\ b\neq 1}} b\mu(q/b)S_d(\vec{\mathbf{a_0}},\vec{\mathbf{m_0}})$$

where $A=\frac{\varphi(q)}{4m!n!}(\frac{\pi}{q})^{m+n+2}$, $\vec{\mathbf{a_0}}= (b;a,1)$, $\vec{\mathbf{m_0}}= (0;m,n)$.

For every real $\alpha>0$, let J_α be the Jordan's totient function defined for all positive integer n by :

$$J_\alpha(n) := n^\alpha\sum_{m\mid n}\frac{\mu(m)}{m^\alpha},$$

where μ is the Mobius function. Since the arithmetical function $J_\alpha(n)/n^\alpha$ is multiplicative, we can write

$$J_\alpha(n) = n^\alpha\prod_{\substack{p\mid n\\ p \text{ prime}}}\left(1-\frac{1}{p^\alpha}\right),\ \text{see [10, p.11,p.219]}.$$

For $\alpha=1$, this is, of course, Euler's function φ.
For $a_1 = \ldots = a_d = 1$ from Theorem 4.0.2 and Theorem 2.3.1, we obtain the following theorem

Theorem 4.0.4. *Let q be an integer ≥ 2. Let d be an integer ≥ 1 and $\vec{\mathbf{m}}= (m_1,\ldots,m_d)$ a d-tuple of positive integers such that the number $M := d+\mid\vec{\mathbf{m}}\mid$ is even. Then*

i) if $(m_1,\ldots,m_d)\neq(0,\ldots,0)$ we have

$$\sum_{(\chi_1,\ldots,\chi_d)}^{*}\prod_{i=1}^{d}L(m_i+1,\chi_i) \;=\; D_q(\vec{\mathbf{m}})\Big(\sum_{j_0=1}^{M/2}\Big(\sum_{\substack{j_1,\ldots,j_d\geq 0\\ j_1+\cdots+j_d=M/2-j_0\\ j_i=0 \text{ ou } \geq(m_i+1)/2}}\prod_{i=1}^{d}A_{i,j_i}\Big)\frac{B_{2j_0}}{(2j_0)!}J_{2j_0}(q)\Big)$$

where

$$D_q(\vec{\mathbf{m}}) = (-1)^{M/2} 2^M A_q(\vec{\mathbf{m}}).$$

ii) for $(m_1, \dots, m_d) = (0, \dots, 0)$ *we have*

$$\sum_{(\chi_1,\dots,\chi_d)}^{*} \prod_{i=1}^{d} L(1,\chi_i) = D_q(\vec{0})\Big(2^{-d}\varphi(q) - \sum_{j_0=1}^{d/2} \big(\sum_{\substack{j_1,\dots,j_d \geq 0 \\ j_1+\dots+j_d = d/2 - j_0}} \prod_{i=1}^{d} \frac{B_{2j_i}}{(2j_i)!}\big) \frac{B_{2j_0}}{(2j_0)!} J_{2j_0}(q)\Big).$$

where $D_q(\vec{0}) = (-1)^{d/2} \left(\frac{\pi}{q}\right)^d \varphi(q)^{d-1}$.

As an immediate consequence, taking $d = 2, m_1 = m_2 = 0$ we obtain a sensitive improvement of Louboutin, Liu and Zhang results [5, 6, 14, 7].

Theorem 4.0.5. *Let m and n be two positive integers having same parity. Then*

- *If $(m,n) \neq (1,1)$, we have*

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \\ \chi(-1)=(-1)^m}} L(m,\chi)L(n,\overline{\chi}) = \frac{1}{2}(-1)^{\frac{m+n}{2}} \left(\frac{2\pi}{q}\right)^{m+n} (M_1 + M_2 + M_3)$$

where

$$\begin{aligned} M_1 &= \frac{B_{m+n}}{(m+n)!} J_{m+n}(q), \\ M_2 &= \frac{(-1)^{m-1}}{(n-1)!m!} \sum_{j=1}^{[m/2]} \binom{m}{2j} \frac{B_{m+n-2j}}{m+n-2j} B_{2j} J_{2j}(q), \\ M_3 &= \frac{(-1)^{n-1}}{(m-1)!n!} \sum_{j=1}^{[n/2]} \binom{n}{2j} \frac{B_{m+n-2j}}{m+n-2j} B_{2j} J_{2j}(q). \end{aligned}$$

- *If $m = n = 1$, we have*

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \\ \chi(-1)=-1}} |L(1,\chi)|^2 = \frac{\pi^2}{6} \frac{\varphi(q)}{q^2} \Big(q \prod_{\substack{p|q \\ p \text{ prime}}} (1 + \frac{1}{p}) - 3 \Big).$$

Proof. For the proof we refer to [2]. □

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ABDELMEJID BAYAD. DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ D'EVRY VAL D'ESSONNE, BD. F. MITTERRAND, 91025 EVRY CEDEX, FRANCE,

E-mail address: abayad@maths.univ-evry.fr